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Maximally entangled mixed states made easy

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We show that it is possible to achieve maximally entangled mixed states of two qubits from the singlet state via the action of local non-trace-preserving quantum channels. Moreover, we present a simple, feasible linear optical implementation of one of such channels.

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I. INTRODUCTION

In a recent paper, Ziman and Bužek have demonstrated that it is impossible to transform the singlet state of two qubits in a maximally entangled mixed state (MEMS), via local maps $\mathcal{E} \otimes \mathcal{I}$ [1]. Such maps describe the action of quantum channels $\mathcal{C}_\mathcal{E}$ acting on a single qubit of the initial singlet state. When a channel is local, that is when it acts on a single qubit, the corresponding map is subjected to some restrictions. This can be easily understood in the following way: Let Alice and Bob be two spatially separated observer who can make measurements on qubits a and b, respectively, and let ρ_in and ρ_out denote the density matrices describing the two-qubit quantum state before and after the channel, respectively. In absence of any causal connection between Alice and Bob, special relativity demands that Bob cannot detect via any type of local measurement the presence of the channel $\mathcal{C}_\mathcal{E}$ in the path of photon a. Since the physics of qubit b is described to Bob by the reduced density matrix $\rho_\text{out}^\text{B} = \text{Tr}\rho_\text{out}|_a$, the locality constraint can be written as

$$\rho_\text{out}^\text{B} = \rho_\text{in}^\text{B}. \tag{1}$$

If we write explicitly the map $\mathcal{E} \otimes \mathcal{I}$ as

$$\rho_\text{in} \mapsto \rho_\text{out} = \sum_\mu \left(A_\mu \otimes I\right) \rho_\text{in} \left(A_\mu^\dagger \otimes I\right), \tag{2}$$

then Eq. (1) becomes

$$\sum_{k,l} (\rho_\text{in})_{li,kj} \Big(\sum_\mu A_\mu^\dagger A_\mu\Big)_{kl} = \sum_k (\rho_\text{in})_{ki,kj}, \tag{3}$$

which implies the *trace-preserving* condition on the local map $\mathcal{E} \otimes \mathcal{I}$:

$$\sum_\mu A_\mu^\dagger A_\mu = I. \tag{4}$$

Local maps that do not satisfy Eq. (4) are classified as *non-physical*, and are not investigated in Ref. [1].

In this paper we show that under certain circumstances, it may be meaningful to consider the action of non-trace-preserving maps, as well. In particular, we give two simple examples of local non-trace-preserving maps that generate maximally entangled mixed states of two qubits from the singlet state. Two-qubit MEMS states may exist in two subclasses usually denoted as MEMS I and MEMS II [2]. In Section II, we furnish an explicit representation for two maps $\mathcal{M}$ and $\mathcal{K}$ that generate MEMS I and II states, respectively. In Sec. III a feasible linear optical implementation of the quantum channel $\mathcal{C}_\mathcal{M}$ corresponding to the map $\mathcal{M}$ is given. In Sec. IV we introduce an all-unitary linear optical model for $\mathcal{C}_\mathcal{M}$ and, via a rigorous QED treatment, we show how the "non-physical" map $\mathcal{M}$ arises in a natural manner. Finally, we draw our conclusions in Sec. V.

II. NON-TRACE-PRESERVING MAPS

In this section we introduce two non-trace-preserving maps $\mathcal{M}$ and $\mathcal{K}$ that generate MEMS I and II states, respectively, from an initial singlet state of two qubits.

A. MEMS I map $\mathcal{M}$

Let ρ_in represent the initial state of a single qubit that is transformed under the action of the map $\mathcal{M}$ as: $\rho_\text{in} \mapsto \rho_\text{out}$, where

$$\rho_\text{out} = \sum_{\mu=0}^{3} \mathbf{M}_\mu \rho_\text{in} \mathbf{M}_\mu^\dagger, \tag{5}$$

and $\mathbf{M}_0 = \mathbf{0} = \mathbf{M}_1$,

$$\mathbf{M}_2 = \sqrt{2(1-p)} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \mathbf{M}_3 = \sqrt{p} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{6}$$

where $2/3 \leq p \leq 1$. This map is *not* trace-preserving nor unital, since

$$\sum_{\mu=0}^{3} \mathbf{M}_\mu^\dagger \mathbf{M}_\mu = \sum_{\mu=0}^{3} \mathbf{M}_\mu \mathbf{M}_\mu^\dagger = \begin{pmatrix} 2-p & 0 \\ 0 & p \end{pmatrix} \neq \mathbf{I}_2, \tag{7}$$

where $\mathbf{I}_n$ denotes the $n \times n$ identity matrix. The apparent non-physical nature of this map can be displayed if we use the Pauli matrices to rewrite

$$\mathbf{M}_2 = \sqrt{\frac{1-p}{2}} (\mathbf{I}_2 + \boldsymbol{\sigma}_z), \qquad \mathbf{M}_3 = -i\sqrt{p}\,\boldsymbol{\sigma}_y, \tag{8}$$

and substitute Eq. (8) into Eq. (5) to obtain

$$\rho_{\rm out} = \frac{1}{2}\big[(1-p)\rho_{\rm in} + 2p\,\boldsymbol{\sigma}_y\rho_{\rm in}\boldsymbol{\sigma}_y + (1-p)\,\boldsymbol{\sigma}_z\rho_{\rm in}\boldsymbol{\sigma}_z + (1-p)\,\{\rho_{\rm in},\boldsymbol{\sigma}_z\}\big], \tag{9}$$

where the anti-commutator term ($\{a,b\} = ab+ba$) is clearly responsible for non conservation of the trace.

Now, let us consider the map $\mathcal{M}$ as representative of the local quantum channel $\mathcal{C}_\mathcal{M}$ acting on a single qubit belonging to an entangled pair prepared in the initial state $\rho_{\rm in}$ (note that *now* $\rho_{\rm in}$ denotes a two-qubit state, therefore it is represented by a 4×4 matrix). The two-qubit map $\mathcal{M}$ can be written as

$$\rho_{\rm out} = \sum_{\mu=0}^{3}\mathbf{N}_\mu\rho_{\rm in}\mathbf{N}_\mu^\dagger, \tag{10}$$

where $\mathbf{N}_\mu = \mathbf{M}_\mu\otimes\mathbf{I}_2$, namely $\mathbf{N}_0 = \mathbf{0} = \mathbf{N}_1$, and

$$\mathbf{N}_2 = \sqrt{2(1-p)}\begin{pmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&0&0\\ 0&0&0&0\end{pmatrix}, \tag{11a}$$

$$\mathbf{N}_3 = \sqrt{p}\begin{pmatrix} 0&0&-1&0\\ 0&0&0&-1\\ 1&0&0&0\\ 0&1&0&0\end{pmatrix}. \tag{11b}$$

Let $|\phi^-\rangle = (|01\rangle - |10\rangle)/\sqrt{2}$ be the two-qubit input singlet state represented by the density matrix $\rho_{\rm s}$:

$$\rho_{\rm s} = |\phi^-\rangle\langle\phi^-| = \begin{pmatrix} 0&0&0&0\\ 0&\frac{1}{2}&-\frac{1}{2}&0\\ 0&-\frac{1}{2}&\frac{1}{2}&0\\ 0&0&0&0\end{pmatrix}. \tag{12}$$

A straightforward calculation shows that

$$\rho_{\rm I} = \sum_{\mu=0}^{3}\mathbf{N}_\mu\rho_{\rm s}\mathbf{N}_\mu^\dagger = \begin{pmatrix} \frac{p}{2}&0&0&\frac{p}{2}\\ 0&1-p&0&0\\ 0&0&0&0\\ \frac{p}{2}&0&0&\frac{p}{2}\end{pmatrix} \tag{13}$$

which represent a MEMS I state.

B. MEMS II map $\mathcal{K}$

As before, let $\rho_{\rm in}$ represent the initial state of a single qubit that transforms under the action of the map $\mathcal{K}$ as: $\rho_{\rm in}\mapsto\rho_{\rm out}$, where

$$\rho_{\rm out} = \sum_{\mu=0}^{3}\mathbf{K}_\mu\rho_{\rm in}\mathbf{K}_\mu^\dagger, \tag{14}$$

where $\mathbf{K}_1 = \mathbf{0}$, and

$$\mathbf{K}_0 = \sqrt{\frac{2}{3}}\begin{pmatrix} 1&0\\ 0&0\end{pmatrix}, \tag{15a}$$

$$\mathbf{K}_2 = \sqrt{\frac{1}{3}-\frac{p}{2}}\begin{pmatrix} 0&1\\ 1&0\end{pmatrix}, \tag{15b}$$

$$\mathbf{K}_3 = \sqrt{\frac{1}{3}+\frac{p}{2}}\begin{pmatrix} 0&-1\\ 1&0\end{pmatrix}, \tag{15c}$$

and $0\le p\le 2/3$. As in the case of $\mathcal{M}$, this map is not trace-preserving nor unital, since

$$\sum_{\mu=0}^{3}\mathbf{K}_\mu^\dagger\mathbf{K}_\mu = \sum_{\mu=0}^{3}\mathbf{K}_\mu\mathbf{K}_\mu^\dagger = \frac{2}{3}\begin{pmatrix} 2&0\\ 0&1\end{pmatrix} \neq \mathbf{I}_2. \tag{16}$$

A straightforward calculation shows that the *two-qubit* map $\mathcal{K}$ realized by $\mathbf{L}_\mu = \mathbf{K}_\mu\otimes\mathbf{I}_2$ produces MEMS II states when acting upon the singlet state (12):

$$\rho_{\rm II} = \sum_{\mu=0}^{3}\mathbf{L}_\mu\rho_{\rm s}\mathbf{L}_\mu^\dagger = \begin{pmatrix} \frac{1}{3}&0&0&\frac{p}{2}\\ 0&\frac{1}{3}&0&0\\ 0&0&0&0\\ \frac{p}{2}&0&0&\frac{1}{3}\end{pmatrix}, \tag{17}$$

where $\mathbf{L}_1 = \mathbf{0}$, and

$$\mathbf{L}_0 = \sqrt{\frac{2}{3}}\begin{pmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&0&0\\ 0&0&0&0\end{pmatrix}, \tag{18a}$$

$$\mathbf{L}_2 = \sqrt{\frac{1}{3}-\frac{p}{2}}\begin{pmatrix} 0&0&1&0\\ 0&0&0&1\\ 1&0&0&0\\ 0&1&0&0\end{pmatrix}, \tag{18b}$$

$$\mathbf{L}_3 = \sqrt{\frac{1}{3}+\frac{p}{2}}\begin{pmatrix} 0&0&-1&0\\ 0&0&0&-1\\ 1&0&0&0\\ 0&1&0&0\end{pmatrix}. \tag{18c}$$

III. LINEAR OPTICAL IMPLEMENTATION OF THE CHANNEL $\mathcal{C}_\mathcal{M}$

The layout of the experiment we propose to create MEMS I states is illustrated schematically in Fig. 1. Two photons in the singlet state $(|HV\rangle - |VH\rangle)/\sqrt{2}$ emerge from the down-converter. Here H and V are labels for horizontally and vertically polarized photons, respectively. Photon b goes directly to detector D_b, while photon a goes to $\mathcal{C}_\mathcal{M}$ and then to detector D_a. $\mathcal{C}_\mathcal{M}$ is a linear optical two-port device that is illustrated in detail

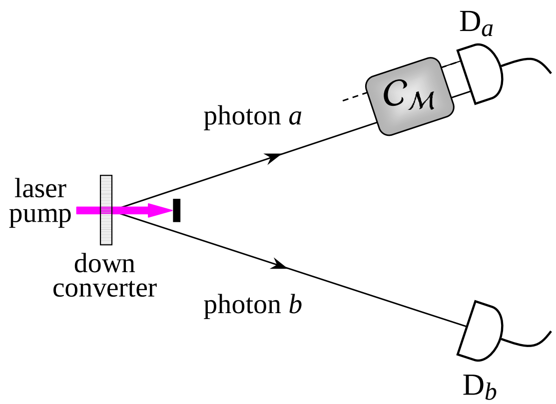

FIG. 1: Sketch of the proposed experimental setup. The box $\mathcal{C}_\mathcal{M}$ represents the quantum channel. A photon from an intense laser pump is split into the pair (a, b) by the down-converter. Detectors D_a and D_b permit a tomographically complete reconstruction of the two-photon quantum state. Further details are given in the text.

in Fig. 2. Supposedly, detector D_a does not distinguish which output port of $\mathcal{C}_\mathcal{M}$ the photon comes from: This is our mechanism to induce decoherence. Photon a enters port **1** and can be either transmitted to path 1 or reflected to path 2 by the 50/50 beam splitter BS; vacuum enters port **2**. Let the square bracket vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ represent the two orthogonal *position* states (or, spatial modes) of a photon travelling in paths 1 and 2, respectively. Analogously, let the parenthesis vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ represent the two *polarization* states of a photon polarized along a horizontal and a vertical direction, respectively. Described in these terms, the 50/50 beam splitter performs a linear transformation restricted to the mode space only; it can be represented by the 2×2 matrix $\mathbf{B}$ as:

$$\mathbf{B} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}, \tag{19}$$

where the relative phase shift of $\pi/2$ between the transmitted and reflected amplitudes ensures unitarity: $\mathbf{B}\mathbf{B}^\dagger = \mathbf{I}_2$. The attenuator A can be simply represented by a scalar function $\exp(-\alpha)$, where $\alpha \geq 0$. The linear polarizer P_H performs a linear transformation restricted to the polarization space only. It can be represented by the projection matrix $\mathbf{H}$ as:

$$\mathbf{H} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{20}$$

Finally, the polarization rotator R can be represented by the orthogonal matrix $\mathbf{R}$ in the polarization space as:

$$\mathbf{R} = \left. \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \right|_{\theta=\frac{\pi}{2}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{21}$$

If with $\mathbf{P}$ and $\mathbf{Q}$ we denote the two complementary mode-space projectors

$$\mathbf{P} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \mathbf{Q} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \tag{22}$$

then the total 4×4 transmission matrix $\mathbf{T}$ representing $\mathcal{C}_\mathcal{M}$ can be written as:

$$\mathbf{T} = \mathbf{R} \otimes \mathbf{P}.\mathbf{B} + e^{-\alpha}\mathbf{H} \otimes \mathbf{Q}.\mathbf{B}, \tag{23}$$

where the low dot " . " denotes the ordinary matrix product.

Let $|\text{in}\rangle$ be the quantum state of a photon entering $\mathcal{C}_\mathcal{M}$ through port 1:

$$|\text{in}\rangle = \begin{pmatrix} \phi_H \\ \phi_V \end{pmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} \equiv |\phi\rangle \otimes |\psi\rangle, \tag{24}$$

where $|\phi_H|^2 + |\phi_V|^2 = 1$. In these terms, the two-mode output state $|\text{out}\rangle$ leaving $\mathcal{C}_\mathcal{M}$ can be written as

$$\begin{aligned} |\text{out}\rangle &= \mathbf{T}|\text{in}\rangle \\ &= \mathbf{R}|\phi\rangle \otimes \mathbf{P}.\mathbf{B}|\psi\rangle + e^{-\alpha}\mathbf{H}|\phi\rangle \otimes \mathbf{Q}.\mathbf{B}|\psi\rangle \\ &= \frac{1}{\sqrt{2}} \left\{ \begin{pmatrix} -\phi_V \\ \phi_H \end{pmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} + i e^{-\alpha} \begin{pmatrix} \phi_H \\ 0 \end{pmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}. \end{aligned} \tag{25}$$

An elementary calculation shows that

$$\begin{aligned} |\text{out}\rangle\langle\text{out}| &= \mathbf{T}|\text{in}\rangle\langle\text{in}|\mathbf{T}^\dagger \\ &= \mathbf{R}|\phi\rangle\langle\phi|\mathbf{R}^\dagger \otimes \mathbf{P}.\mathbf{B}|\psi\rangle\langle\psi|\mathbf{B}^\dagger.\mathbf{P}^\dagger \\ &\quad + e^{-2\alpha}\mathbf{H}|\phi\rangle\langle\phi|\mathbf{H}^\dagger \otimes \mathbf{Q}.\mathbf{B}|\psi\rangle\langle\psi|\mathbf{B}^\dagger.\mathbf{Q}^\dagger \\ &\quad + \big\{ e^{-\alpha}\mathbf{R}|\phi\rangle\langle\phi|\mathbf{H}^\dagger \otimes \mathbf{P}.\mathbf{B}|\psi\rangle\langle\psi|\mathbf{B}^\dagger.\mathbf{Q}^\dagger \\ &\quad + \text{H.c.} \big\}, \end{aligned} \tag{26}$$

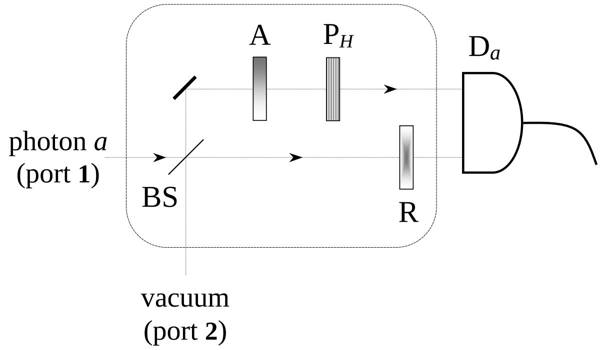

FIG. 2: Detailed scheme of the quantum channel $\mathcal{C}_\mathcal{M}$. BS denotes a 50/50 beam splitter, A is a beam attenuator, P_H is a linear polarizer that selects horizontally polarized photons, and R is a polarization rotator oriented at $\theta = \pi/2$.

where H.c. stands for Hermitian conjugate.

Since, by hypothesis, detector D_a does not distinguish a photon exiting port **1** from a photon exiting port **2**, the *detected* output state can be obtained from $|\mathrm{out}\rangle\langle\mathrm{out}|$ by tracing over the detected but unresolved position states:

$$\mathrm{Tr}\big[|\mathrm{out}\rangle\langle\mathrm{out}|\big] = \frac{1}{2}\left(\mathbf{R}|\phi\rangle\langle\phi|\mathbf{R}^\dagger + e^{-2\alpha}\mathbf{H}|\phi\rangle\langle\phi|\mathbf{H}^\dagger\right), \tag{27}$$

where trivially $\mathrm{Tr}\big[\mathbf{P}.\mathbf{B}|\psi\rangle\langle\psi|\mathbf{B}^\dagger.\mathbf{Q}^\dagger\big] = 0$, and $\mathrm{Tr}\big[\mathbf{P}.\mathbf{B}|\psi\rangle\langle\psi|\mathbf{B}^\dagger.\mathbf{P}^\dagger\big] = 1/2 = \mathrm{Tr}\big[\mathbf{Q}.\mathbf{B}|\psi\rangle\langle\psi|\mathbf{B}^\dagger.\mathbf{Q}^\dagger\big]$. If we define $J_{\mathrm{out}} \equiv \mathrm{Tr}\big[|\mathrm{out}\rangle\langle\mathrm{out}|\big]$ and $\rho_{\mathrm{in}} \equiv |\phi\rangle\langle\phi|$, then we can rewrite Eq. (27) as

$$\begin{aligned} J_{\mathrm{out}} &= \frac{1}{2}\left(\mathbf{R}\rho_{\mathrm{in}}\mathbf{R}^\dagger + e^{-2\alpha}\mathbf{H}\rho_{\mathrm{in}}\mathbf{H}^\dagger\right) \\ &= \frac{1}{2p}\left(\mathbf{M}_2\rho_{\mathrm{in}}\mathbf{M}_2^\dagger + \frac{p\,e^{-2\alpha}}{2(1-p)}\mathbf{M}_3\rho_{\mathrm{in}}\mathbf{M}_3^\dagger\right), \end{aligned} \tag{28}$$

where Eqs. (6), (20), and (21) have been used. If we choose $\alpha = \alpha(p)$ such that

$$\frac{p\,e^{-2\alpha}}{2(1-p)} = 1 \quad \Rightarrow \quad \alpha(p) = -\frac{1}{2}\ln\frac{2(1-p)}{p}, \tag{29}$$

then Eq. (28) can be rewritten as

$$J_{\mathrm{out}} = \frac{1}{2p}\sum_{\mu=0}^{3}\mathbf{M}_\mu\rho_{\mathrm{in}}\mathbf{M}_\mu^\dagger = \frac{1}{2p}\rho_{\mathrm{out}}, \tag{30}$$

where Eq. (5) has been used. Note that $\alpha(p) \geq 0$ for $2/3 \leq p \leq 1$, as expected for an attenuator. Equation (30) shows that the scheme shown in Fig. 2 actually implements the map $\mathcal{M}$. Moreover, from Eqs. (24,28-29) it follows that

$$\begin{aligned} \mathrm{Tr}\left(J_{\mathrm{out}}\right) &= \frac{1}{2}\left[1 + \frac{2(1-p)}{p}|\phi_H|^2\right], \\ \therefore\ \frac{1}{2} &\leq \mathrm{Tr}\left(J_{\mathrm{out}}\right) \leq 1, \end{aligned} \tag{31}$$

for $2/3 \leq p \leq 1$ and $0 \leq |\phi_H| \leq 1$. This means that even in the worst case ($p = 1$) there is still a 50% of probability to detect a photon in our scheme.

IV. RIGOROUS QED TREATMENT

It was pointed out [3] that the map $\mathcal{M}$ corresponds to a *non-physical* quantum channel $\mathcal{C}_{\mathcal{M}}$. Conversely, in the previous section we have shown that a *physical* linear optical implementation of $\mathcal{C}_{\mathcal{M}}$ is actually feasible. The resolution of this apparent paradox lies in the conceptual difference that exists between the "quantum state of two qubits", and the "*measured* quantum state of two qubits". The latter can be reconstructed only after Alice and Bob have performed coincidence measurements [4], that is only *after* they have established a communication and have compared their own experimental results. Therefore, a *measured* MEMS state generated by a local channel does not raise any causality issue. In this spirit, we will soon show how the map $\mathcal{M}$ can be derived from an all-unitary model for the channel $\mathcal{C}_{\mathcal{M}}$ (Fig. 3). Such a unitary channel reduces to a non-unitary one when Alice restricts her measurements to two output ports only (**1** and **2**), leaving the other two (**3** and **4**) undetected. However, note that in principle Alice could use an additional detector $\mathrm{D}_a^{(\mathbf{34})}$ coupled to ports **3** and **4** to generate a "conditional" MEMS state: When a photon pair is created by the down-converter and detector $\mathrm{D}_a^{(\mathbf{34})}$ does not fire, then a conditional MEMS state is being transmitted through the channel.

Let indicate with $a_{i\alpha}$ and b_α the annihilation operators of photons a and b, respectively. Greek indexes $\alpha, \beta, \ldots \in \{0, 1\}$ label *polarization* modes of the field, while Latin indexes $i, j, \ldots \in \{1, 2, 3, 4\}$ label *spatial* modes of the field. The latter modes represent the four paths shown in Fig. 3. Described in these terms, the two-photon input

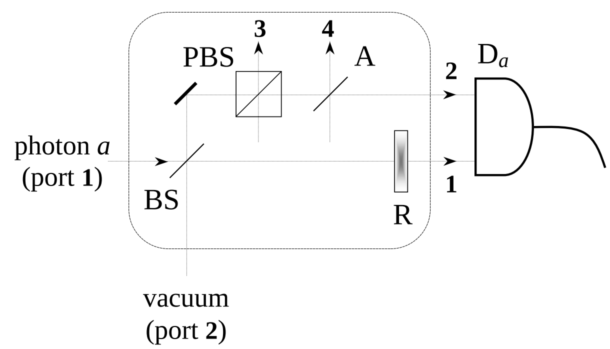

FIG. 3: All-unitary linear optical realization of the quantum channel $\mathcal{C}_{\mathcal{M}}$. PBS is a polarizing beam splitter that transmits only horizontally polarized photons. The beam attenuator A is implemented by a variable-reflectivity beam splitter. The two additional modes **3** and **4** ensure the whole unitary nature of $\mathcal{C}_{\mathcal{M}}$.

singlet state can be written as

$$|\mathrm{in}\rangle = \frac{1}{\sqrt{2}}\left(a_{10}^\dagger b_1^\dagger - a_{11}^\dagger b_0^\dagger\right)|0\rangle, \tag{32}$$

where $|0\rangle$ denotes the vacuum state. Each linear optical element present in the quantum channel shown in Fig. 3, can be represented by a unitary operator U that evolves the state vector $|\psi\rangle$ as

$$|\psi\rangle \mapsto U|\psi\rangle, \tag{33}$$

and the operator X either as

$$X \mapsto U^\dagger X U, \tag{34}$$

or as

$$X \rightarrowtail U X U^\dagger. \tag{35}$$

In particular, if the annihilation operator $a_{i\alpha}$ evolves as

$$a_{i\alpha} \mapsto U^\dagger a_{i\alpha} U = \sum_{j=1}^{4} \sum_{\beta=0}^{1} S_{i\alpha,j\beta} a_{j\beta}, \tag{36}$$

then it is easy to see that

$$a_{i\alpha}^\dagger \rightarrowtail U a_{i\alpha}^\dagger U^\dagger = \sum_{j=1}^{4} \sum_{\beta=0}^{1} S_{j\beta,i\alpha} a_{j\beta}^\dagger. \tag{37}$$

Within this formalism, the beam splitter BS is described by the matrix

$$S_{j\beta,i\alpha} = \mathbf{B}_{ji} \delta_{\beta\alpha} \tag{38}$$

where $\mathbf{B}$ is explicitly given in Eq. (19). This leads to the field operators transformation

$$a_{1\alpha}^\dagger \rightarrowtail \frac{1}{\sqrt{2}} \left(a_{1\alpha}^\dagger + i a_{2\alpha}^\dagger \right), \tag{39}$$

that modifies the input state $|\text{in}\rangle$ to:

$$|\text{in}\rangle \mapsto \frac{1}{2} \left(a_{10}^\dagger b_1^\dagger + i a_{20}^\dagger b_1^\dagger - a_{11}^\dagger b_0^\dagger - i a_{21}^\dagger b_0^\dagger \right) |0\rangle. \tag{40}$$

The effect of the rotator R is very simple:

$$a_{10}^\dagger \rightarrowtail a_{11}^\dagger, \qquad a_{11}^\dagger \rightarrowtail -a_{10}^\dagger, \tag{41}$$

and it changes the two-photon states to

$$|\text{in}\rangle \mapsto \frac{1}{2} \left(a_{11}^\dagger b_1^\dagger + i a_{20}^\dagger b_1^\dagger + a_{10}^\dagger b_0^\dagger - i a_{21}^\dagger b_0^\dagger \right) |0\rangle. \tag{42}$$

Next, the polarizing beam splitter PBS can be described by a 4×4 unitary matrix that couples both spatial and polarization modes, as

$$\begin{pmatrix} a_{20}^\dagger \\ a_{21}^\dagger \\ a_{30}^\dagger \\ a_{31}^\dagger \end{pmatrix} \rightarrowtail \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 1 & 0 \\ 0 & i & 0 & 0 \end{pmatrix} \begin{pmatrix} a_{20}^\dagger \\ a_{21}^\dagger \\ a_{30}^\dagger \\ a_{31}^\dagger \end{pmatrix} \tag{43}$$

As a result of this transformation, the two photon state after the PBS can be written as

$$|\text{in}\rangle \mapsto \frac{1}{2} \left(a_{11}^\dagger b_1^\dagger + i a_{20}^\dagger b_1^\dagger + a_{10}^\dagger b_0^\dagger + a_{31}^\dagger b_0^\dagger \right) |0\rangle. \tag{44}$$

Finally, the attenuator A can be described in a unitary fashion by modelling it as a variable-reflectivity beam splitter such that:

$$a_{2\alpha}^\dagger \rightarrowtail T a_{2\alpha}^\dagger + i R a_{4\alpha}^\dagger, \tag{45}$$

where $0 \leq T \leq 1$, and $T^2 + R^2 = 1$. This last optical element produces the output state $|\text{out}\rangle$, where

$$\begin{aligned} |\text{out}\rangle \;&=\; \frac{1}{2} \Big[\left(a_{11}^\dagger b_1^\dagger + a_{10}^\dagger b_0^\dagger + i T a_{20}^\dagger b_1^\dagger \right) \\ &\quad + \left(a_{31}^\dagger b_0^\dagger - R a_{40}^\dagger b_1^\dagger \right) \Big] |0\rangle \\ &\equiv\; |\psi_{12}\rangle + |\psi_{34}\rangle. \end{aligned} \tag{46}$$

In Eq. (46) $|\psi_{ij}\rangle$ denotes the two-photon state restricted to the pair of modes (i, j), and $\langle \psi_{12} | \psi_{34} \rangle = 0$. Since each transformation performed by each linear optical element present in the quantum channel is unitary, the output state $|\text{out}\rangle$ is still normalized: $\langle \text{out} | \text{out} \rangle = 1$. Now we can trace over the spatial degrees of freedom in the usual way obtaining:

$$\rho = \text{Tr} \left[|\text{out}\rangle\langle\text{out}| \right] \equiv \rho_{12} + \rho_{34}, \tag{47}$$

where

$$\rho_{12} = \frac{1}{4} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & T^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \tag{48}$$

and

$$\rho_{34} = \frac{1}{4} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 - T^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{49}$$

The *total* density matrix ρ has trace equal to one, while the *truncated* density matrices ρ_{12} and ρ_{34} have nonunit trace:

$$\text{Tr}\left(\rho_{12}\right) = \frac{1}{2} \left(1 + T^2/2 \right) \equiv \frac{1}{2p}; \tag{50a}$$

$$\text{Tr}\left(\rho_{34}\right) = \frac{1}{2} \left(1 - T^2/2 \right) \equiv 1 - \frac{1}{2p}. \tag{50b}$$

This simple result shows that each truncated density matrix *cannot* be generated by a trace-preserving map. In particular, we easily recover our result Eq. (13) by dividing Eq. (48) by Eq. (50a). This is the goal of the present section.

To conclude, it may be instructive to calculate separately the *reduced* density matrices $\text{Tr}\rho_{ij}|_f$, obtained by tracing over the degrees of freedom of photon f, where $(i, j) \in \{(1, 2), (3, 4)\}$ and $f = a, b$:

$$\begin{aligned} \text{Tr}\rho_{12}|_a &= \frac{1}{4} \begin{pmatrix} 1 & 0 \\ 0 & 1 + T^2 \end{pmatrix}, \;\; \text{Tr}\rho_{12}|_b = \frac{1}{4} \begin{pmatrix} 1 + T^2 & 0 \\ 0 & 1 \end{pmatrix}, \\ \text{Tr}\rho_{34}|_a &= \frac{1}{4} \begin{pmatrix} 1 & 0 \\ 0 & 1 - T^2 \end{pmatrix}, \;\; \text{Tr}\rho_{34}|_b = \frac{1}{4} \begin{pmatrix} 1 - T^2 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned} \tag{51}$$

From these results we learn that

$$\text{Tr}\rho_{12}|_a + \text{Tr}\rho_{34}|_a = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \text{Tr}\rho_{\text{s}}|_a, \tag{52a}$$

$$\text{Tr}\rho_{12}|_b + \text{Tr}\rho_{34}|_b = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \text{Tr}\rho_{\text{s}}|_b, \tag{52b}$$

that is, when *all* spatial modes of the two photons are properly accounted for, locality requirements are fully satisfied.

V. CONCLUSIONS

Equations (11), (13), (17), (18), and (30), are the main results of our preliminary work on generation and measurement of maximally entangled mixed states. In this paper we have shown how it is possible to generate both MEMS I and II two-qubit states from the singlet state by using only *local*, non-trace-preserving quantum channels. Moreover, we provided for the scheme of a simple linear optical experimental setup for the generation of photonic MEMS I states. Such a scheme, which exploit spatial degrees of freedom of the photons to induce decoherence, is currently being tested in our laboratory.

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